Review : Differential Geometry of Curves and Surfaces

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Abstract

The brief review of **do Carmo**'s *Differential Geometry of Curves and Surfaces.*

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1 Curves

1.1 Introduction

1.2 Parametrized Curves

Definition. A parametrized differentiable curve is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

1.3 Regular Curves; Arc Length

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized differentiable curve. For each $t \in I$ where $\alpha'(t) \neq 0$, there is a well-defined straight line, which contains the point $\alpha(t)$ and the vector $\alpha'(t)$. This line is called the *tangent line* to α at t.

Definition. A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

Given $t \in I$, the *arc length* of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, from the point t_0, is by definition

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| \, dt = \int_{t_0}^{t} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

It can happen that the parameter t is already the arc length measured from some point. In this case, $ds/dt = 1 = |\alpha'(t)|$.

1.4 The Vector Product in $\mathbb{R}^3$

Two ordered bases $e = \{e_i\}$ and $f = \{f_i\}$ of an n-dimensional vector space V have the *same orientation* if the matrix of change of basis ha positive determinant. We denote this relation by $e \sim f$.

1.5 The Local Theory of Curves Parametrized by Arc Length

Definition. Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $|\alpha''(s)| = k(s)$ is called the *curvature* of α at s.
Notice: $\alpha'(s)$ has unit length, since α is a curve parametrized by arc length s.

At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$. $n(s)$ is normal to $\alpha'(s)$ and is called the *normal vector* at s. The plane determined by the unit tangent and normal vectors, $\alpha'(s)$ and $n(s)$, is called the *osculating plane* at s. At points where $k(s) = 0$, the normal vector (and therefore the osculating plane) is not defined.

The unit vector $b(s) = t(s) \wedge n(s)$ is normal to the osculating plane and will be called the *binormal vector* at s.

Definition. Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0$, $s \in I$. The number $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the *torsion* of α at s.

$$\begin{cases} \alpha''(s) = k(s)n(s) \\ \alpha'(s) = t(s) \end{cases} \qquad \begin{cases} b(s) = t(s) \wedge n(s) \\ b'(s) = t(s) \wedge n'(s) = \tau(s)n(s) \end{cases}$$

The trihedron formed by three orthogonal unit vectors $t(s)$, $n(s)$, $b(s)$ is referred to as the *Frenet trihedron* at s. And the Frenet Formulas :

$$\begin{cases} t' = kn \\ n' = -kt - \tau b \\ b' = \tau n \end{cases}$$

The tb plane is called the *rectifying plane* and the nb plane the *nomal plane*. The lines wichi contain $n(s)$ and $b(s)$ and pass through $\alpha(s)$ are called the *principla normal* and the *binormal*.

Fundamental Theorem of the Local Theory of Curves Given differentiable functions $k(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$, satisfying the same conditions, differs from α by a rigid motion; that is, there exists an orthogonal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that $\bar{\alpha} = \rho \circ \alpha + c$.

1.6 The Local Canonical Form †

$$\alpha(s) - \alpha(0) = (s - \frac{k^2}{6}s^3)t + (\frac{k}{2}s^2 + \frac{k'}{6}s^3)n - \frac{s^3}{6}k\tau b + R$$

$$x(s) = s - \frac{k^2}{6}s^3 + R_x \qquad y(s) = \frac{k}{2}s^2 + \frac{k'}{6}s^3 + R_y \qquad z(s) = -\frac{k\tau}{6}s^3 + R_z$$

The representation above is called the *local canonical form* of α, in a neighborhood of $s = 0$.

1.7 Global Properties of Plane Curves †

A differentiable function on a closed interval $[a, b]$ is the *restriction* of a differentiable function defined on an open interval containing $[a, b]$.

A closed plane curve is a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that α and all its derivatives agree at a and b; that

$$\alpha(a) = \alpha(b), \qquad \alpha'(a) = \alpha'(b), \qquad \alpha''(a) = \alpha''(b)$$

The curve α is *simple* if it has no further self-intersections.

The Isoperimetric Inequality

Theorem. Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then

$$l^2 - 4\pi A \geq 0$$

and equality holds if and only if C is a circle.

The Four-Vertex Theorem

A *vertex* of a regular plane curve $\alpha : [a, b] \to \mathbb{R}^2$ is a point $t \in [a, b]$ where $k'(t) = 0$.

Theorem. A simple closed convex curve has at least four vertices.

The Cauchy-Crofton Formula

Let C be a regular plane curve with length l. The measure of the set of straight lines (counted with multiplicities) which meet C is equal to $2l$.

2 Regular Surfaces

2.1 Introduction

In contrast to the treatment of curves in chapter 1, regular surfaces are defined as sets rather than maps.

2.2 Regular Surfaces; Inverse Images of Regular Values

Definition 1. A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable. This means that if we write $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, the functions x y z have continuous partial derivatives of all orders in U.
2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.
3. (The regularity condition) For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The mapping $\mathbf{x}$ is called a parametrization of a system of (local) coordiantes in (a neighborhood of) p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

Proposition 1. If $f : U \to R$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is the subset of $\mathbb{R}^3$ given by $(x, y, f(x,y))$ for $(x,y) \in U$, is a regular surface.

Definition 2. Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical value is called a regular value of F.

Proposition 2. If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Proposition 3. Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable funciton which has one of the following three forms: $z = f(x,y)$, $y = g(x,z)$, $x = h(y,z)$.

Proposition 4. Let $p \in S$ be a point of a regular surface S and let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{x}(U) \subset S$ such that conditions 1 and 3 of Def. 1 hold. Assume that $\mathbf{x}$ is one-to-one. Then $\mathbf{x}^{-1}$ is continuous.

2.3 Change of Parameters; Differential Functions on Surfaces

Proposition 1 (Change of Parameters). Let p be a point of a regular surface S, and let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the "change of coordinates" $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism; that is h is differentiable and has a differentiable inverse h^{-1}.

Definition 1. Let $f : V \subset S \to R$ be a function defined in an open subset V of a regular surface S. Then f is said to be *differentiable at* $p \in V$ if for some parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$ is differentiable in V if it is differentiable at all points of V.

Two regular surfaces S_1 and S_2 are *diffeomorphic* if there exists a differentiable map $\phi : S_1 \to S_2$ with a differentiable inverse $\phi^{-1} : S_2 \to S_1$. Such a ϕ is called a *diffeomorphism* from S_1 to S_2.

Since S can be entirely covered by similar parametrizations, it follows that S is a regular surface wihich is called a surface of revolution. The curve C is called the generating curve of S, and the z axis is the rotation axis of S. The circles described by the points of C are called the parallels of S, and the various positions of C on S are called the meridians of S.

Definition 2. A parametrized surface $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable map $\mathbf{x}$ from an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$. The set $\mathbf{x}(U) \subset \mathbb{R}^3$ is called the trace of $\mathbf{x}$. $\mathbf{x}$ is *regular* if the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for all $q \in U$. A point $p \in U$ where $d\mathbf{x}_q$ is not one-to-one is called a singular point of $\mathbf{x}$.

Proposition 2. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface and let $q \in U$. Then there exists a neighborhood V of q in $\mathbb{R}^2$ such that $\mathbf{x}(V) \subset \mathbb{R}^3$ is a regular surface.

2.4 The Tangent Plane; the Differential of a Map

Proposition 1. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S at $\mathbf{x}(q)$.

The plane $d\mathbf{x}_q(\mathbb{R}^2)$, which passes through $\mathbf{x}(q) = p$, does not depend on the parametrization $\mathbf{x}$. This plane will be called the tangent plane to S at p and will be denoted by $T_p(S)$. The choice of the parametrization $\mathbf{x}$ determines a basis $\{(\partial\mathbf{x}/\partial u(q), \partial\mathbf{x}/\partial v(q))\}$ of $T_p(S)$, called the basis associated to $\mathbf{x}$. Conveniently, $\partial\mathbf{x}/\partial u = \mathbf{x}_u$ and $\partial\mathbf{x}/\partial v = \mathbf{x}_v$.

The coordinates of a vector $w \in T_p(S)$ in the basis associated to a parametrization $\mathbf{x}$ are determined as follows. w is the velocity vector $\alpha'(0)$ of a curve $\alpha = \mathbf{x} \circ \beta$, where $\beta : (-\epsilon, \epsilon) \to U$ is given by $\beta(t) = (u(t), v(t))$, with $\beta(0) = q = \mathbf{x}^{-1}(p)$. Thus,

$$\begin{aligned} w = \alpha'(0) = \frac{d}{dt}(\mathbf{x} \circ \beta)(0) = d\mathbf{x}_q(\beta(0)) \cdot \beta'(0) &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} \\ &= \mathbf{x}_u(q)u'(0) + \mathbf{x}_v(q)v'(0) \end{aligned}$$

thus in the basis $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$, w has coordinates $(u'(0), v'(0))$.

Proposition 2. In the discussion above, given w, the vector $\beta'(0)$ does not depend on the choice of α. The map $d\phi_p : T_p(S_1) \to T_{\phi(p)}(S_2)$ defined by $d\phi_p(w) = \beta'(0)$ is linear.

Proposition 3. If S_1 and S_2 are regular surfaces and $\phi : U \subset S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ such that the differential $d\phi_p$ of ϕ at $p \in U$ is an isomorphism, then ϕ is a local diffeomorphism at p.

By fixing a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ at $p \in S$, we can make a definite choice of a unit normal vector at each point $q \in \mathbf{x}(U)$ by the rule

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q)$$

2.5 The First Fundamental Form; Area

$$I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0$$

Definition 1. The quadratic form I_p on $T_p(S)$ defined above, is called the *first fundamental form* of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

$$\begin{aligned} I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\ &= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle \\ &= E(u')^2 + 2Fu'v' + G(v')^2 \end{aligned}$$

$$E(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p \qquad F(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p \qquad G(u_0, v_0) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

2.6 Orientation of Surfaces †

The bases $\{\mathbf{x}_u, \mathbf{x}_v\}$ and $\{\bar{\mathbf{x}}_{\bar{u}}, \bar{\mathbf{x}}_{\bar{v}}\}$ determine the same orientation of $T_p(S)$ if and only if the Jacobian

$$\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}$$

of the coordinate change is positive.

Definition 1. A regular surface S is called *orientable* if it is possible to cover it with a family of coordinate neighborhoods in such a way that if a point $p \in S$ belongs to two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of such a family is called an *orientation* of S, and S in this case is called *oriented*. If such a choice is not possible, the surface is called *nonorientable*.

Proposition 1. A regular surface $S \subset \mathbb{R}^3$ is orientable if and only if there exists a differentiable field of unit normal vectors $N : S \to \mathbb{R}^3$ on S.

Proposition 2. If a regular surface is given by $S = \{(x, y, z) \in \mathbb{R}^3; f(x, y, z) = a\}$, where $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable and a is a regular value of f, then S is orientable.

Orientation is a global property, in the sense that it involves the whole surface.

2.7 A Characterization of Compact Orientable Surfaces †

That an orientable surface in $\mathbb{R}^3$ is the inverse image of a regular value of some differentiable function, is true.

Proposition 1. Let S be a regular surface and $\mathbf{x} : U \to S$ be a parametrizatino of a neighborhood of a point $p = \mathbf{x}(u_0, v_0) \in S$. Then there exists a neighborhood $W \subset \mathbf{x}(U)$ of p in S

and a number $\epsilon > 0$ such that the segments of the normal lines passing through points $q \in W$, with center at q and length 2ϵ, are disjoint (that is, W has a tubular neighborhood.)

Proposition 2. Assume the existence of a tubular neighborhood $V \subset \mathbb{R}^3$ of an orientable surface $S \subset \mathbb{R}^3$, and choose an orientation for S. Then the function $g : V \to R$, defined as the oriented distance from a point of V to the foot of the unique normal line passing through this point, is differentiable and has zero as a regular value.

Let A be a subset of $\mathbb{R}^3. We say that$p $\in \mathbb{R}^3$ is a *limit point* of A if every neighborhood of p in $\mathbb{R}^3$ contains a point of A distinct from p. A is said to be *closed* if it contains all its limit points. A is *bounded* if it is contained in some ball of $\mathbb{R}^3$. If A is closed and bounded, it is called a *compact set*.

Property 1 (Bolzano-Weierstrass). Let $A \subset \mathbb{R}^3$ be a compact set. Then every infinite subset of A has at least one limit point in A.

Property 2 (Heine-Borel). Let $A \subset \mathbb{R}^3$ be a compact set and $\{U_\alpha\}$ be a family of open sets of A such that $\cup_\alpha U_\alpha = A$. Then it is possible to choose a finite number $U_{k_1}, \cdots, U_{k_n}$ of U_α such that $\cup U_{k_i} = A$, $i = 1, \cdots, n$.

Property 3 (Lebesgue). Let $A \subset \mathbb{R}^3$ be a compact set and $\{U_\alpha\}$ a family of open sets of A such that $\cup_\alpha U_\alpha = A$. Then there exists a number $\delta > 0$ (the Lebesgue number of the family $\{U_\alpha\}$) such that whenever two points $p, q \in A$ are at a distance $d(p, q) < \delta$, then p and q belong to some U_α.

Proposition 3. Let $S \subset \mathbb{R}^3$ be a regular, compact, orientable surface. Then there exists a number $\epsilon > 0$ such that whenever $p, q \in S$ the segments of the normal lines of length 2ϵ, centered in p and q, are disjoint (that is, S has a tubular neighborhood).

Theorem. Let $S \subset \mathbb{R}^3$ be a regular, compact, orientable surface. Then there exists a differentiable function $g : V \to R$, defined in an open set $V \subset \mathbb{R}^3$, with $V \supset S$ (precisely a tubular neighborhood of S), which has zero as a regular value and is such that $S = g^{-1}(0)$.

2.8 A Geometric Definition of Area †

Proposition. Let $\mathbf{x} : U \to S$ be a coordinate system in a regular surface S and let $R = \mathbf{x}(Q)$ be a bounded region of S contained in $\mathbf{x}(U)$. Then E has an area given by

$$A(R) = \iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv$$

3 The Geometry of the Gauss Map

3.1 Introduction

We shall try to measure how rapidly a surface S pulls away from the tangent plane $T_p(S)$ in a neighborhood of a point $p \in S$.

3.2 The Definition of the Gauss Map and Its Fundamental Properties

We shall say that s regular surface is orientable if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field N is called an *orientation* of S.

Definition 1. Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; |x^2 + y^2 + z^2 = 1\}$$

The map $N : S \to S^2$, thus defined, is called the *Gauss map* of S.

Proposition 1. The differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.

Definition 2. The quadratic form II_p, defined in $T_p(S)$ by $II_p(v) = -\langle dN_p(v), v\rangle$ is called the *second fundamental form* of S at p.

Definition 3. Let C be a regular curve in S passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N\rangle$, where n is the normal vector to C and N is the normal vector to S at p. The number $k_n = k\cos\theta$ is then called the *normal curvature* of $C \subset S$ at p.

Proposition 2 (Meusnier). All curves lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvatures.

Definition 4. The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the *principal curvatures* at p; the corresponding directions, that is, the directions given by the eigenvectors e_1, e_2, are called *principal directions* at p.

Definition 5. If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p, then C is said to be a *line of curvature* of S.

Proposition 3 (Olinde Rodrigues). A necessary and sufficient condition for a connected regular curve C on S to be a line of curvature of S is that

$$N'(t) = \lambda(t)\alpha'(t)$$

for any parametrization $\alpha(t)$ of C, where $N(t) = N \circ \alpha(t)$ and $\lambda(t)$ is a differentiable function of t. In this case, $-\lambda(t)$ is the (principal) curvature along $\alpha'(t)$.

Definition 6. Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the *Gaussian curvature K* of S at p. The negative of half of the trace of dN_p is called the *mean curvature H* of S at p.

$$K = k_1 k_2 \qquad H = \frac{k_1 + k_2}{2}$$

Definition 7. A point of a surface S is called

1. Elliptic if $det(dN_p) > 0$.
2. Hyperbolic if $det(dN_p) < 0$.
3. Parabolic if $det(dN_p) = 0$, with $dN_p \neq 0$.
4. Planar if $dN_p = 0$.

Definition 8. If at $p \in S$, $k_1 = k_2$, then p is called an *umbilical point* of S; in particular, the planar points ($k_1 = k_2 = 0$) are umbilical points.

Proposition 4. If all points of a connected surface S are umbilical points, then S is either contained in a sphere or in a plane.

Definition 9. Let p be a point in S. An *asymptotic direction* of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.

Definition 10. Let p be a point on a surface S. Two nonzero vectors $w_1, w_2 \in T_p(S)$ are *conjugate* if $\langle dN_p(w_1), w_2 \rangle = \langle w_1, dN_p(w_2) \rangle = 0$. Two directions r_1, r_2 at p are conjugate if a pair of nonzero vectors w_1, w_2 parallel to r_1 and r_2, respectively, are conjugate.

3.3 The Gauss Map in Local Coordinates

Some concepts related to the local behavior of the Gauss map. We shall obtain the expressions of the second fundamental form and of the differential of the Gauss map in a coordinate system.

All parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ considered in this section are assumed to be compatible with the orientation N of S; that is, in $\mathbf{x}(U)$.

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$$

The tangent vector to $\alpha(t)$ at p is $\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$ and

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'$$

since N_u and N_v belong to $T_p(S)$, we may write

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

thus dN is given by the matrix (a_{ij}). It is symmetric if $\{\mathbf{x}_u, \mathbf{x}_v\}$ is an orthonormal basis.

The expression of the second fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is given by

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle = -\langle N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle \\ &= e(u')^2 + 2fu'v' + g(v')^2 \end{aligned}$$

$$e = -\langle N_u, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_{uu} \rangle \qquad f = -\langle N_v, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_{uv} \rangle \qquad g = -\langle N_v, \mathbf{x}_v \rangle = \langle N, \mathbf{x}_{vv} \rangle$$

$$K = det(a_{ij}) = \frac{eg - f^2}{EG - F^2}$$

$$H = \frac{k_1 + k_2}{2} = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$

$$k = H \pm \sqrt{H^2 - K}$$

Proposition 1. Let $p \in S$ be an elliptic point of a surface S. Then there exists a neighborhood V of p in S such that all points in V belong to the same side of the tangent plane $T_p(S)$. Let $p \in S$ be a hyperbolic point. Then in each neighborhood of p there exist points of S in both sides of $T_p(S)$.

Notice that in any neighborhood of such a parabolic point there exist points in both sides of the tangent plane.

A connected regular curve C in the coordinate neighborhood of $\mathbf{x}$ is an asymptotic curve if and only if for any parametrization $\alpha(t) = \mathbf{x}(u(t), v(t))$, $t \in I$, of C we have $II(\alpha'(t)) = 0$, for all $t \in I$, that is, if and only if (*the differential equation of the asymptotic curves*)

$$e(u')^2 + 2fu'v' + g(v')^2 = 0$$

A connected regular curve C in the coordinate neighborhood of $\mathbf{x}$ is a line of curvature if and only if for any parametrization $\alpha(t) = \mathbf{x}(u(t), v(t))$ of C, $t \in I$, we have

$$dN(\alpha'(t)) = \lambda(t)\alpha'(t)$$

by eliminating λ, we obtain the *differential equation of the lines of curvature*,

$$\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Proposition 2. Let p be a point of a surface S such that the Gaussian curvature $K(p) \neq 0$, and let V be a connected neighborhood of p where K does not change sign. Then

$$K(p) = \lim_{A \to 0} \frac{A'}{A}$$

where A is the area of a region $B \subset V$ containing p, A' is the area of the image of B by the Gauss map $N : S \to S^2$, and the limit is taken through a sequence of regions B_n that converges to p, in the sense that any sphere around p contains all B_n, for n sufficiently large.

3.4 Vector Fields †

3.5 Ruled Surfaces and Minimal Surfaces †

Ruled Surfaces

Given a one-parameter family of lines $\{\alpha(t), w(t)\}$, the parametrized surface

$$\mathbf{x}(t, v) = \alpha(t) + vw(t), \qquad t \in I, \qquad v \in R$$

is called the *ruled surface* generated by the family $\{\alpha(t), w(t)\}$. The lines L_t are called the *rulings*, and the curve $\alpha(t)$ is called a *directrix* of the surface $\mathbf{x}$.

Minimal Surfaces

A regular parametrized surface is called *minimal* if its mean curvature vanishes everywhere.

Proposition 1. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a regular parametrized surface and let $D \subset U$ be a bounded domain in U. Then $\mathbf{x}$ is minimal if and only if $A'(0) = 0$ for all such D and all normal variations of $\mathbf{x}(\bar{D})$.

The *mean curvature vector* defined by $\mathbf{H} = HN$

A regular parametrized surface $\mathbf{x} = \mathbf{x}(u,v)$ is said to be *isothermal* if $\langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$ and $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$.

Proposition 2. Let $\mathbf{x} = \mathbf{x}(u,v)$ be a regular parametrized surface and assume that $\mathbf{x}$ is isothermal. Then

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2\lambda^2 \mathbf{H}$$

where $\lambda^2 = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$.

We say that f is *harmonic* in U if $\Delta f = (\partial^2 f/\partial u^2) + (\partial^2 f/\partial v^2) = 0$.

Corollary : Let $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ be a parametrized surface and assume that $\mathbf{x}$ is isothermal. Then $\mathbf{x}$ is minimal if and only if its coordinate functions x, y, z are harmonic.

Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface and define complex functions ϕ_1, ϕ_2, ϕ_3 by

$$\phi_1(\zeta) = \frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v} \qquad \phi_2(\zeta) = \frac{\partial y}{\partial u} - i\frac{\partial y}{\partial v} \qquad \phi_3(\zeta) = \frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v}$$

where x, y, z are the component functions of $\mathbf{x}$.

Lemma. $\mathbf{x}$ is isothermal if and only if $\phi_1^2 + \phi_2^2 + \phi_3^2 \equiv 0$. If this last condition is satisfied, $\mathbf{x}$ is minimal if and only if ϕ_1, ϕ_2, ϕ_3 are analytic functions.

Theorem (Osserman). Let $S \subset \mathbb{R}^3$ be a regular, closed (as a subset of $\mathbb{R}^3$) minimal surface in $\mathbb{R}^3$ which is not a plane. The the image of the Gauss map $N : S \to S^2$ is dense in the sphere S^2 (that is , arbitrarily close to any point of S^2 there is a point of $N(S) \subset S^2$).

4 The Intrinsic Geometry of Surfaces

4.1 Introduction

4.2 Isometries; Conformal Maps

Definition 1. A diffeomorphism $\phi : S \to \bar{S}$ is an *isometry* if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have

$$\langle w_1, w_2 \rangle_p = \langle d\phi_p(w_1), d\phi_p(w_2) \rangle_{\phi(p)}$$

The surfaces S and $\bar{S}$ are then said to be *isometric*.

Definition 2. A map $\phi : V \to \bar{S}$ of a neighborhood V of $p \in S$ is a *local isometry* at p if there exists a neighborhood $\bar{V}$ of $\phi(p) \in \bar{S}$ such that $\phi : V \to \bar{V}$ is an isometry. If there exists a local isometry into $\bar{S}$ at every $p \in S$, the surface S is said to be *locally isometric* to $\bar{S}$. S and $\bar{S}$ are locally isometric if S is locally isometric to $\bar{S}$ and $\bar{S}$ is locally isometric to S.

Proposition 1. Assume the existence of parametrizations $\mathbf{x} : U \to S$ and $\bar{\mathbf{x}} : U \to \bar{S}$ such that $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ in U. Then the map $\phi = \bar{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \bar{S}$ is a local isometry.

Definition 3. A diffeomorphism $\phi : S \to \bar{S}$ is called a *conformal map* if for all $p \in S$ and all $v_1, v_2 \in T_p(S)$ we have

$$\langle d\phi_p(v_1), d\phi_p(v_2) \rangle = \lambda^2(p) \langle v_1, v_2 \rangle_p$$

where λ^2 is a nowhere-zero differentiable function on S; the surfaces S and $\bar{S}$ are then said to be *conformal*. A map $\phi : V \to \bar{S}$ of a neighborhood V of $p \in S$ into $\bar{S}$ is a *local conformal map* at p if there exists a neighborhood $\bar{V}$ of $\phi(p)$ such that $\phi : V \to \bar{V}$ is a conformal map. If for each $p \in S$, there exists a local conformal map at p, the surface S is said to be locally conformal to $\bar{S}$.

Proposition 2. Let $\mathbf{x} : U \to S$ and $\bar{\mathbf{x}} : U \to \bar{S}$ be parametrizations such that $E = \lambda^2 \bar{E}$, $F = \lambda^2 \bar{F}$, $G = \lambda^2 \bar{G}$ in U, where λ^2 is a nowhere-zero differentiable function in U. Then the map $\phi = \bar{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \bar{S}$ is a local conformal map.

Theorem. Any two regular surfaces are locally conformal.

4.3 The Gauss Theorem and the Equations of Compatibility

Proceeding with the analogy with curves, we are going to assign to each point of a surface a trihedron given by the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and N. We introduce the coefficients Γ_{ij}^k

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l} \right)$$

All geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.

Theorema Egregium (Gauss). The Gaussian curvature K of a surface is invariant by local isometries.

Gauss formula :

$$(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2 = -E\frac{eg-f^2}{EG-F^2} = -EK$$

Mainardi-Codazzi equation :

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2$$
$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2$$

The Gauss formula and the Mainardi-Codazzi equations are known under the name of *compatibility equations* of the theory of surfaces.

Theorem (Bonnet). Let E, F, G, e, f, g be differentiable functions, defined in an open set $V \subset \mathbb{R}^3$, with $E > 0$ and $G > 0$. Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that $EG - F^2 > 0$. Then, for every $q \in V$ there exists a neighborhood $U \subset V$ of q and a diffeomorphism $\mathbf{x} : U \to \mathbf{x}(U) \subset \mathbb{R}^3$ such that the regular surface $\mathbf{x}(U) \subset \mathbb{R}^3$ has E, F, G and e, f, g as coefficients of the first and second fundamental forms, respectively. Furthermore, if U is connected and if

$$\bar{\mathbf{x}} : U \to \bar{\mathbf{x}} \subset \mathbb{R}^3$$

is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation ρ in $\mathbb{R}^3$ such that $\bar{\mathbf{x}} = T \circ \rho \circ \mathbf{x}$.

4.4 Parallel Transport; Geodesics

Definition 1. Let w be a differentiable vector field in an open set $U \subset S$ and $p \in U$. Let $y \in T_p(S)$. Consider a parametrized curve

$$\alpha : (-\epsilon, \epsilon) \to U$$

with $\alpha(0) = p$ and $\alpha'(0) = y$, and let $w(t)$, $t \in (-\epsilon, \epsilon)$, be the restriction of the vector field w to the curve α. The vector obtained by the normal projection of $(dw/dt)(0)$ onto the plane $T_p(S)$ is called the covariant derivative at p of the vector field w relative to the vector y. This covariant derivative is denoted by $(Dw/Dt)(0)$ or $(D_y w)(p)$.

Definition 2. A parametrized curve $\alpha : [0, l] \to S$ is the restriction to $[0, l]$ of a differentiable mapping of $(0 - \epsilon, l + \epsilon)$, $\epsilon > 0$, into S. If $\alpha(0) = p$ and $\alpha(l) = q$, we say that α joins p to q. α is regular if $\alpha'(t) \neq 0$ for $t \in [0, l]$.

$$\frac{Dw}{dt} = ()$$

Definition 3. Let $\alpha : I \to S$ be a parametrized curve in S. A vector field w along α is a correspondence that assigns to each $t \in I$ a vector

$$w(t) \in T_{\alpha(t)}(S)$$

The vector field w is differentiable at $t_0 \in I$ if for some parametrization $\mathbf{x}(u, v)$ in $\alpha(t_0)$ the components $a(t), b(t)$ of $w(t) = a\mathbf{x}_u + b\mathbf{x}_v$ are differentiable functions of t at t_0. w is differentiable in I if it is differentiable for every $t \in I$.

Definition 4. Let w be a differentiable vector field along $\alpha : I \to S$. The expression of $(Dw/dt)(t)$, $t \in I$, is well defined and is called the *covariant derivative* of w at t.

Definition 5. A vector field w along a parametrized curve $\alpha : I \to S$ is said to be *parallel* if $Dw/dt = 0$ for every $t \in I$.

Proposition 1. Let w and v be parallel vector fields along $\alpha : I \to S$. Then $\langle w(t), v(t) \rangle$ is constant. In particular, $|w(t)|$ and $|v(t)|$ are constant, and the angle between $v(t)$ and $w(t) is constant.$

Proposition 2. Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\alpha(t)$, with $w(t_0) = w_0$.

Definition 6. Let $\alpha : I \to S$ be a parametrized curve and $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Let w be the parallel vector field along α, with $w(t_0) = w_0$. The vector $w(t_1)$, $t_1 \in I$, is called the *parallel transport* of w_0 along α at the point t_1.

Definition 7. A map $\alpha : [0, l] \to S$ is a parametrized piecewise regular curve if α is continuous and there exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$$

of the interval $[0, l]$ in such a way that the restriction $\alpha| [t_i, t_{i+1}]$, $i = 0, \cdots, k$, is a parametrized regular curve. Each $\alpha| [t_i, t_{i+1}]$ is called *a regular arc* of α.

Definition 8. A nonconstant, parametrized curve $\gamma : I \to S$ is said to be *geodesic* at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t; that is

$$\frac{D\gamma'(t)}{dt} = 0$$

γ is a *parametrized geodesic* if it is geodesic for all $t \in I$.

Definition 8a. A regular connected curve C in S is said to be a geodesic if, for every $p \in C$, the parametrization $\alpha(s)$ of a coordinate neighborhood of p by the arc length s is a parametrized geodesic; that is, $\alpha'(s)$ is a parallel vector field along $\alpha(s)$.

Definition 9. Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S. Since $w(t)$, $t \in I$, is a unit vector field, $(dw/dt)(t)$ is normal to $w(t)$, and therefore

$$\frac{Dw}{dt} = \lambda(N \wedge w(t))$$

The real number $\lambda = \lambda(t)$, denoted by $[Dw/dt]$, is called the *algebraic value of the covariant derivative* of w at t.

Definition 10. Let C be an oriented regular curve contained on an oriented surface S, and let $\alpha(s)$ be a parametrization of C, in a neighborhood of $p \in S$, by the arc length s. The algebraic value of the covariant derivative $[D\alpha'(s)/ds] = k_g$ of $\alpha'(s)$ at p is called the *geodesic curvature* of C at p.

Lemma 1. Let a and b be differentiable functions in I with $a^2 + b^2 = 1$ and φ_0 be such that $a(t_0) = \cos\varphi_0$, $b(t_0) = \sin\varphi_0$. THen the differentiable function

$$\varphi = \varphi_0 \int_{t_0}^{t} (ab' - ba')dt$$

is such that $\cos\varphi(t) = a(t)$, $\sin\varphi(t) = b(t)$, $t \in I$, and $\phi(t_0) = \varphi_0$.

Lemma 2. Let v and w be two differentiable vector fields along the curve $\alpha : I \to S$, with $|w(t)| = |v(t)| = 1$, $t \in I$. Then

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\varphi}{dt}$$

where φ is one of the differentiable determinations of the angle from v to w, as given above.

Proposition 3. Let $\mathbf{x}(u, v)$ be an orthogonal parametrization (that is $F = 0$) of a neighborhood of an oriented surface S, and $w(t)$ be a differentiable field of unit vectors along the curve $\mathbf{x}(u(t), v(t))$ .Then

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right\} + \frac{d\varphi}{dt}$$

where $\varphi(t)$ is the angle from $\mathbf{x}_0$ to $w(t)$ in the given orientation.

Proposition 4 (Liouville). Let $\alpha(s)$ be a parametrization by arc length of a neighborhood of a point $p \in S$ of a regular oriented curve C on an oriented surface S. Let $\mathbf{x}(u, v)$ be an orthogonal parametrization of S in p and $\phi(s)$ be the angle that $\mathbf{x}_u$ makes with $\alpha'(s)$ in the given orientation. Then

$$k_g = (k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \frac{d\varphi}{ds}$$

where $(k_g)_1$ and $(k_g)_2$ are the geodesic curvatures of the coordinate curves $v = const.$ and $u = const.$ respectively.

Proposition 5. Given a point $p \in S$ and a vector $w \in T_p(S)$, $w \neq 0$, there exist an $\epsilon > 0$ and a unique parametrized geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$, $\gamma'(0) = w$.

4.5 The Gauss-Bonnet Theorem and its Applications

$$\sum_{i=1}^{3} \varphi_i - \pi = \iint_T K d\sigma$$

Let $\alpha : [0, l] \to S$ be a continuous map from the closed interval $[0, l]$ into the regular surface S. We say that α is a simple, closed, piecewise regular parametrized curve if

1. $\alpha(0) = \alpha(l)$.

2. $t_1 \neq t_2$, $t_1, t_2 \in [0, l)$ implies that $\alpha(t_1) \neq \alpha(t_2)$.

3. There exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$$

of $[0, l]$ such that α is differentiable and regular in each $[t_i, t_{i+1}]$, $i = 0, \cdots, k$.

Theorem (of Turning Tangents). With the above notation

$$\sum_{i=0}^{k}(\varphi_i(t_{i+1}) - \varphi_i(t_i)) + \sum_{i=0}^{k}\theta_i = \pm 2\pi$$

where the sign plus or minus depends on the orientation of α.

Gauss-Bonnet Theorem (Local). Let $\mathbf{x} : U \to S$ be an orthogonal parametrization (that is $F = 0$), of an oriented surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an open disk and $\mathbf{x}$ is compatible with the orientation of S. Let $R \subset \mathbf{x}(U)$ be a simple region of S and let $\alpha : I \to S$ be such that $\partial R = \alpha(I)$. Assume that α is positively oriented, parametrized by arc length s, and let $\alpha(s_0), \cdots, \alpha(s_k)$ and $\theta_0, \cdots, \theta_k$ be, respectively, the vertices and the external angles of α. Then

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)ds + \iint_R K d\sigma + \sum_{i=0}^{k}\theta_i = 2\pi$$

where $k_g(s)$ is the geodesic curvature of the regular arcs of α and K is the Gaussian curvature of S.

Proposition 1. Every regular region of a regular surface admits a triangulation.

Proposition 2. Let S be an oriented surface and $\mathbf{x}_\alpha$, $\alpha \in A$, a family of parametrizations compatible with the orientation of S. Let $R \subset S$ be a regular region of S. Then there is a triangulation $\mathfrak{J}$ of R such that every triangle $T \in \mathfrak{J}$ is contained in some coordinate neighborhood of the family $\mathbf{x}_\alpha$. Furthermore, if the boundary of every triangle of $\mathfrak{J}$ is positively oriented, adjacent triangles determine opposite orientations in the common edge.

Proposition 3. If $R \subset S$ is a regular region of a surface S, the Euler-Poincaré characteristic does not depend on the triangulation of R. It is convenient, therefore, to denote it by $\chi(R)$.

Proposition 4. Let $S \subset \mathbb{R}^3$ be a compact connected surface; then one of the values $2, 0, -2, \cdots, -2n, \cdots$ is assumed by the Euler-Poincaré characteristic $\chi(S)$. Furthermore, if $S' \subset \mathbb{R}^3$ is another compact surface and $\chi(S) = \chi(S')$, then S is homeomorphic to S'.

In other words, every compact connected surface $S \subset \mathbb{R}^3$ is homeomorphic to a sphere with a certain number g of handles. The number

$$g = \frac{2 - \chi(S)}{2}$$

is called the *genus* of S.

Proposition 5. With the above notation, the sum

$$\sum_{i=1}^{k}\iint_{\mathbf{x}_i^{-1}(T_i)} f(u_i, v_i)\sqrt{E_iG_i - F_i^2}du_idv_i$$

does not depend on the triangulation $\mathfrak{I}$ or on the family $\mathbf{x}_i$ of parametrizations of S.

GLOBAL GAUSS-BONNET THEOREM. Let $R \subset S$ be a regular region of an oriented surface and let $C_1, \cdots, C_n$ be the closed, simple, piecewise regular curves which form the boundary ∂R of R. Suppose that each C_i is positively oriented and let $\theta_1, \cdots, \theta_p$ be the set of all external angles of the curves $C_1, \cdots, C_n$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi\chi(R)$$

where s denotes the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_i.

Corollary 1. If R is a simple region of S, then

$$\sum_{i=1}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi$$

Corollary 2. Let S be an orientable compact surface; then

$$\iint_S K d\sigma = 2\pi\chi(S)$$

Poincaré's theorem The sum of the indices of a differentiable vector field v with isolated singular points on a compact surface S is equal to the Euler-Poincaré characteristic of S.

4.6 The Exponential Map. Geodesic Polar Coordinates

Lemma 1. If the geodesic $\gamma(t, v)$ is defined for $t \in (-\epsilon, \epsilon)$, then the geodesic $\gamma(t, \lambda v)$, $\lambda \neq 0 \in \mathbb{R}$, is defined for $t \in (-\epsilon/\lambda, \epsilon/\lambda)$, and $\gamma(t, \lambda v) = \gamma(\lambda t, v)$.

Proposition 1. Given $p \in S$ there exists an $\epsilon > 0$ such that $\exp_p$ is defined and differentiable in the interior B_ϵ of a disk of radius ϵ of $T_p(S)$, with center in the origin.

Proposition 2. $\exp_p : B_\epsilon \subset T_p(S) \to S$ is a diffeomorphism in a neighborhood $U \subset B_\epsilon$ of the origin 0 of $T_p(S)$.

Proposition 3. Let $\mathbf{x} : U - l \to V - L$ be a system of geodesic polar coordinates (ρ, θ). Then the coefficients $E = E(\rho, \theta)$, $F = F(\rho, \theta)$, and $G = G(\rho, \theta)$ of the first fundamental form satisfy the conditions

$$E = 1 \qquad F = 0 \qquad \lim_{\rho \to 0} G = 0 \qquad \lim_{\rho \to 0} (\sqrt{G})_\rho = 1$$

Theorem (Minding). Any two regular surfaces with the same constant Gaussian curvature are locally isometric. More precisely, let S_1, S_2 be two regular surfaces with the same constant curvature K. Choose points $p_1 \in S_1, p_2 \in S_2$, and orthonormal basis $e_1, e_2 \in T_{p_1}(S_1)$, $f_1, f_2 \in T_{p_2}(S_2)$. Then there exist neighborhoods V_1 of p_1, V_2 of p_2 and an isometry $\psi : V_1 \to V_2$ such that $d\psi(e_1) = f_1$, $d\psi(e_2) = f_2$.

Proposition 4. Let p be a point on a surface S. Then, there exists a neighborhood $W \subset S$ of p such that if $\gamma : I \to W$ is a parametrized geodesic with $\gamma(0) = p, \gamma(t_1) = q, t_1 \in I$, and $\alpha : [0, t_1] \to S$ is a parametrized regular curve joining p to q, we have

$$l_\gamma \leq l_\alpha$$

where l_α denotes the length of the curve α. Moreover, if $l_\gamma = l_\alpha$, then the trace of γ coincides with the trace of α between p and q.

Proposition 5. Let $\alpha : I \to S$ be a regular parametrized curve with a parameter proportional to arc length. Suppose that the arc length of α between any two points $t, \tau \in I$, is smaller than or equal to the arc length of any regular parametrized curve joining $\alpha(t)$ to $\alpha(\tau)$. Then α is a geodesic.

4.7 Further Properties of Geodesics. Convex Neighborhoods †

Theorem 1. Given $p \in S$ there exist numbers $\epsilon_1 > 0, \epsilon_2 > 0$ and a differentiable map

$$\gamma : (-_2, \epsilon_2) \times B_{\epsilon_1} \to S, \qquad B_{\epsilon_1} \subset T_p(S)$$

such that for $v \in B_{\epsilon_1}$, $v \neq 0$, $t \in (-_2, \epsilon_2)$ the curve $t \to \gamma(t, v)$ is the geodesic of S with $\gamma(0, v) = p$, $\gamma'(0, v) = v$, and for $v = 0, \gamma(t, 0) = p$.

Theorem 1a. Given $p \in S$, there exist positive numbers $\epsilon, \epsilon_1, \epsilon_2$ and a differentiable map

$$\gamma : (-_2, \epsilon_2) \times \mathfrak{U} \to S$$

where

$$\mathfrak{U} = (q, v); q \in B_\epsilon(p), v \in B_{\epsilon_1}(0) \subset T_q(S)$$

such that $\gamma(t, q, 0) = q$, and for $v \neq 0$ the curve

$$t \to \gamma(t, q, v), \qquad t \in (-_2, \epsilon_2)$$

is the geodesic of S with $\gamma(0, q, v) = q, \gamma'(0, q, v) = v$.

Proposition 1. Given $p \in S$ there exist a neighborhood W of p in S and a number $\delta > 0$ such that for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0) \subset T_q(S)$ and $\exp_q(B_\delta(0)) \supset W$; that is, W is a normal neighborhood of all its points.

Proposition 2. Let $\alpha : I \to S$ be a parametrized, piecewise regular curve such that in each regular arc the parameter is proportional to the arc length. Suppose that the arc length between any two of its points is smaller than or equal to the arc length of any parametrized regular curve joining these points. Then α is a geodesic; in particular, α is regular everywhere.

Proposition 3. For each point $p \in S$ there exists a positive number ϵ with the following property: If a geodesic $\gamma(t)$ is tangent to the geodesic circle $S_r(p)$, $r < \epsilon$, at $\gamma(0)$, then, for $t \neq 0$ small, $\gamma(t)$ is outside $B_r(p)$.

Proposition 4 (Existence of Convex Neighborhoods). For each point $p \in S$ there exists a number $c > 0$ such that $B_c(p)$ is convex; that is, any two points of $B_c(p)$ can be joined by a unique minimal geodesic in $B_c(p)$.

5 Global Differential Geometry

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